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Fingering in Hele-Shaw cells

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A large class of explicit solutions for Hele-Shaw flow with a free surface is presented. The results are valid when surface-tension effects in the plane of the cell are negligible. Most of the solutions given produce fingers, both in channel flow and on a growing air bubble. Possible behaviour of these fingers is described, and a qualitative comparison with published experimental results is made.

1. Introduction

The phenomenon of 'fingering' in a porous medium occurs when, in the flow of two immiscible fluids, a fluid of higher viscosity is displaced by one of lower viscosity. The interface between the two fluids is unstable to perturbations of certain wavelengths, and long 'fingers' of the less-viscous fluid penetrate the more-viscous one, leaving, in some cases, a considerable proportion of the latter behind. This instability is sometimes known as the Saffman–Taylor instability, and is of interest in models of secondary recovery of oil from underground reservoirs in which the oil is forced out of the ground by injection of a less viscous fluid, usually water.

In view of the difficulties of performing experiments on oil reservoirs, there has been considerable interest in the use of a Hele-Shaw cell as an experimental analogue of two-dimensional flow in porous media (Paterson 1981, and references therein). The equations we obtain from the simplest mathematical models of the two situations are, apart from a scaling, the same, and the Hele-Shaw cell is easier to use as an experimental tool. This is one motivation for studying Hele-Shaw flow; it is, in addition, an interesting problem in fluid dynamics in its own right, and the simple mathematical model which we shall use is a member of an important class of Stefan-type moving-boundary problems which are currently receiving much attention as models of processes such as crystal growth from a supersaturated solution, solidification of supercooled liquids, and freezing or melting of a dilute binary alloy (Langer 1980).

In this paper we give some explicit solutions for finger development in one-phase flow in a Hele-Shaw cell (that is, a flow in which the viscosity of the less-viscous fluid is so small that it may be ignored). These are obtained by a complex-variable method, and they are valid under certain simplifying assumptions about the boundary conditions applied on the interface between the two fluids, primarily under the assumption that the effects of surface tension in the plane of the cell are small. The equations of motion and complex-variable method are briefly reviewed in §2, together with the linear stability of a typical moving boundary.

Two configurations are considered. In the first, set out in §3, the growth of a small perturbation of a planar interface in a horizontal cell in the shape of an infinitely long channel with parallel walls is studied as fluid is extracted from far upstream. Of particular interest here is that we can find infinitely many exact solutions of which some, starting with small perturbations which may be arbitrarily close to each other in some suitable measure, evolve through quite different intermediate stages into the same large-time, travelling-wave solution near their tip, a solution which was first studied by Saffman & Taylor (1958); others, however, can be shown to 'blow up' in finite time via a cusp in their moving boundary.

In the second configuration, described in §4, we present explicit solutions which represent the growth of radial fingers from an initially almost circular bubble of air in an infinite cell filled with viscous fluid; the bubble expands as air is injected. A brief comparison is made with the experimental results of Paterson (1981).

These solutions are given by conformal maps depending on a set of real functions of time which appear as coefficients in the mapping function. A necessary condition for the map to represent a solution is that these functions satisfy a complicated set of ordinary differential equations, and in §5 we discuss a connection between these equations and the shape of the moving interface which allows the solutions to the differential equations to be written down immediately.

Finally it may be helpful to give a justification for studying these solutions. The model we use is a simple one, and as a model of flow in a real Hele-Shaw cell it has shortcomings; the chief of these is probably the neglect of surface-tension effects in the plane of the cell at the moving boundary. We can see the results of this neglect in the linear stability result (equation (2.9)), and even more dramatically in the widespread occurrence of finite-time blow-up via cusp formation in the moving boundary of this simple model (see §3). Nevertheless, there are still some solutions to the simple model which do not have finite-time blow-up. It is our belief that the inclusion of a small surface tension term at the moving boundary will, in some way which is not fully understood (but which is doubtless related to the improvement in linear stability thus achieved (equation (2.10))), act as a filter to exclude those solutions of the simple model which have finite-time blow-up. That is, we believe that the solutions which are seen in practice when the surface tension is small are in some measure 'close' to solutions of the simple model which do not blow up, and that this is one good reason for studying the latter. Lastly they have intrinsic interest in the context of the other moving-boundary problems mentioned earlier.

2. Equations of motion; linear stability; application of complex variables

Hele-Shaw flow takes place when a viscous fluid moves slowly between two fixed parallel plates (which we shall assume to be horizontal), separated by a thin gap. The mean flow is two-dimensional, and the mean velocity components in the plane of the cell are given, in suitable dimensionless form, by

$$(u, v) = -\nabla p, \tag{2.1}$$

where the pressure $p(x, y, t)$ satisfies

$$\nabla^2 p = 0 \tag{2.2}$$

in the fluid region, which we denote by $\Omega(t)$.

At a rigid boundary inside the cell, the flow is tangential. At an interface between fluid and air, which is sufficiently thin to be treated as a curve $\partial\Omega(t)$ in the (x, y)-plane, we shall take

$$p = 0 \tag{2.3}$$

and, from mass conservation,

$$-\frac{\partial p}{\partial n} = V_n, \tag{2.4}$$

where $\boldsymbol{n}$ is the outward normal to $\partial\Omega$ and V_n the normal velocity of a point on $\partial\Omega$. This equation can also be written

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\partial p}{\partial t} - \nabla p \cdot \nabla p = 0. \tag{2.5}$$

These questions are discussed by Saffman & Taylor (1958). When the surface-tension forces at $\partial\Omega$ are large in the lateral (x, y)-direction, (2.3) may be replaced by

$$p = \gamma\kappa, \tag{2.6}$$

where γ is a dimensionless surface-tension coefficient and κ the lateral curvature of $\partial\Omega$; (2.5) is no longer valid in this case, and (2.4) must be used. We shall only briefly mention situations in which $\partial\Omega$ is highly curved and where (2.6) must be invoked even though γ is small.

The conditions at infinity are that for a growing bubble, with extraction rate $Q > 0$,

$$p \sim -\frac{Q}{2\pi} \ln (x^2+y^2)^{\frac{1}{2}} \quad \text{as } x^2+y^2 \to \infty, \tag{2.7}$$

while for flow in a channel with parallel walls $y = \pm\pi$,

$$p \sim -Vx \quad \text{as } x \to \infty, \tag{2.8}$$

where the fluid is extracted uniformly with speed V far upstream, at a rate $Q = 2\pi V$. Lastly, we shall assume that, at $t = 0$, $\partial\Omega$ is a given analytic curve.

We can examine the linear stability of a planar interface $x = Vt$ (with fluid in $x > Vt$) to perturbations with wavenumber n. The result is that a perturbed interface $x = Vt + \epsilon^{\sigma t} \sin ny$ has, by the linear theory, a growth rate

$$\sigma = |n| V \tag{2.9}$$

if we use condition (2.3), whereas it is

$$\sigma = |n| V - \gamma |n|^3 \tag{2.10}$$

if we use (2.6) (Saffman & Taylor 1958). Equation (2.9) implies that in the simple model with (2.3), perturbations of *all* wavelengths of a receding interface are unstable, with short-wavelength perturbations growing the fastest. We shall see later that receding interfaces can in fact develop cusps in finite time, and the solution can fail to exist after that time; indeed, any solution which does not develop cusps is 'close' (in the sense of polynomial approximation at $t = 0$) to infinitely many which do.

We shall present explicit solutions for both the configurations mentioned above. In each case, using the complex-variable method of Galin (1945) and Richardson (1972), we map the disk $|\zeta| < 1$ conformally onto the fluid region by a mapping

$$z = x + \mathrm{i}y = f(\zeta; t), \tag{2.11}$$

where $f(\zeta; 0)$ is known from $\Omega(0)$. We require that $|\zeta| = 1$ goes onto $\partial\Omega(t)$, and that the origin is mapped onto the sink at infinity. For flow in a channel, we also require that the walls $y = \pm\pi$ are mapped onto a branch cut along the negative real-ζ axis, while for flow outside a bubble we require $\arg z \sim \arg(1/\zeta)$ at infinity. The pressure in the ζ-plane is just $(Q/2\pi)\ln|\zeta|$, and an application of the rules for change of variable in (2.8) gives

$$\operatorname{Re}\left(\zeta\frac{\partial f}{\partial \zeta}\frac{\overline{\partial f}}{\partial t}\right) = -\frac{Q}{2\pi} \tag{2.12}$$

as a boundary condition for the unknown analytic function f on $|\zeta| = 1$ (Galin 1945). Any f with the appropriate singularities and which satisfies (2.12) gives a solution to the Hele-Shaw flow; in §§3 and 4 we present several classes of such solutions.

3. Channel flow

The most celebrated Hele-Shaw solutions for channel flow are the travelling-wave fingers of Saffman & Taylor (1958). We shall presently discuss examples which tend, for large time, to these solutions; but first we show that any solution which does not develop cusps can, at $t = 0$, be approximated by infinitely many polynomial solutions which do.

We shall assume that at $t = 0$ the fluid occupies the region to the right of a simple analytic curve joining $y = -\pi$ to $y = \pi$ and perpendicular to both. A suitable form for $f(\zeta;t)$ is then

$$z = -\ln\zeta + h(\zeta;t), \tag{3.1}$$

in which h is analytic in $|\zeta| < 1$ for each t, and is real on the real-ζ axis. (This predicates that the finger is symmetric, but an asymmetric finger can be made symmetric by reflection in one of the channel walls.)

It is required in addition that $|\mathrm{d}f/\mathrm{d}\zeta| > 0$ in $|\zeta| < 1$ and that f is 1–1 ($\Omega(t)$ does not overlap itself).

Suppose now that we have a solution, such as the ones given later in this section, which exists for all t. Then the function corresponding to this solution can, at $t = 0$, be uniformly approximated as closely as we please by infinitely many polynomials of the form

$$\sum_{n=0}^{N} a_n(0)\,\zeta^n \quad (a_n(0) \text{ real}): \tag{3.2}$$

consequently we can approximate $\Omega(0)$ for this flow arbitrarily closely with a polynomial for h in (3.1).

It can easily be shown that taking h to be a polynomial in ζ whose coefficients are functions of t does in fact give an explicit solution to (2.12): one obtains $N+1$ differential equations for the $N+1$ coefficients. It can also be shown that if $N \geqslant 1$ this solution cannot exist for all t; this follows by integrating the first and last of these equations and obtaining a contradiction from a balance of terms as $t \to \infty$ (see Howison 1986*a* for a similar argument in the radial geometry).

We do not intend to discuss this blow-up at length since it cannot occur in practice. Here we merely mention that it typically results in a 3/2 power cusp (with infinite fluid velocities) in the moving boundary. There are other possible ways in which blow-up can occur, and more details are given by Howison, Lacey & Ockendon (1985) and Howison (1986*a*). Instead, we pass on to consider solutions which exist for all time without this blow-up; but we bear in mind the large class of blow-up solutions in which they are embedded.

The travelling-wave-finger solution of Saffman & Taylor (1958) is given, in our notation, by

$$z = \frac{Vt}{\lambda} - \ln\zeta + 2(1+\lambda)\ln\tfrac{1}{2}(1+\zeta), \tag{3.3}$$

in which the constant λ is the ratio of the width of the finger at $x = -\infty$ to the channel width, and where the nose of the finger is at the origin at $t = 0$. The feature of this solution that has excited the most interest is that the mathematical solution exists for any $0 < \lambda \leqslant 1$, and more information is needed to specify λ. If we regard channel flow as being one period of a periodic array of fingers without walls, then the spacing of the fingers may be taken as 2π (in the absence of surface tension there is no natural lengthscale in the problem), and so there is a one-parameter family of possible large-time travelling-wave outcomes of the fingering instability of a planar interface. (The celebrated question of which values of λ will be seen in practice is discussed by Vandenbroeck 1983, Maclean & Saffman 1981, and in references therein.) It was pointed out by Saffman (1959) that there is an explicit solution, obtainable from (3.3) by allowing some of the coefficients to be functions of time,

$$z = \mathrm{d}(t) - \ln\zeta + 2(1-\lambda)\ln\tfrac{1}{2}(1+a(t)\,\zeta), \tag{3.4}$$

which, if $0 < a(0) \ll 1$, represents the growth of an initially almost-flat interface with a small harmonic perturbation, $x = 2a(0)(1-\lambda)\cos y + O(a(0)^2)$, through intermediate stages until as $t \to \infty$, $a \to 1$ and $d \sim Vt/\lambda$, and we retrieve asymptotically the travelling-wave solution (3.3) (see figure 1*a*). Now the constructive way in which (3.3) is derived (Saffman & Taylor 1958) shows that it is the only possible form for a steady, travelling wave, single, symmetric finger, and so any unsteady solution with a travelling-wave limit must tend to (3.3). The evolving finger (3.4) shows that there is at least one solution having the steady finger as its asymptotic form, and it may now be asked whether Saffman's solutions (3.4) are the only such. We can answer this question in the negative by presenting a large class of explicit solutions of this type. They are simple generalizations of (3.4), given by

$$z = \mathrm{d}(t) - \ln\zeta + \frac{1}{N}\sum_{n=1}^{N} 2(1-\lambda_n)\ln\tfrac{1}{2}(1+a_n(t)\,\zeta), \tag{3.5}$$

where for the moment $0 \leqslant a_n < 1$, and where λ, the asymptotic width of the finger, is $(1/N)\sum_{n=1}^{N}\lambda_n$, which must lie between 0 and 1 (note that if $N > 1$, some of the λ_n may be greater than 1).

Substituting (3.5) into (2.12) and equating coefficients we obtain exactly $N+1$ differential equations for $d, a_1, \ldots, a_N$, whose solution with suitable initial values $d(0), a_1(0), \ldots, a_N(0)$ determines uniquely the evolution of the finger. The $O(1)$ coefficient gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda d + \frac{1}{N}\sum_{n=1}^{N}(1-\lambda_n)\ln a_n\right) = V, \tag{3.6}$$

which is simply conservation of mass, while the remaining N equations are complicated and will not be given here, since they can be integrated easily using the procedure of §5. Thus we merely give the result, a set of N transcendental equations linking d and the a_n: for $n = 1, \ldots, N$,

$$d - \ln a_n + \frac{1}{N}\sum_{m=1}^{N} 2(1-\lambda_m)\ln(1-a_m a_n) = \text{constant}. \tag{3.7}$$

When $N = 1$, (3.5) is the Saffman solution (3.4). For $N > 1$ however, and for certain values of the λ_n and the $a_n(0) \in (0,1)$ we shall see that we can get strikingly different behaviour near the base of the finger, although the tip is always, for large time, asymptotic to (3.3).

Before giving specific examples, it is helpful to find the shape of the base of the fingers as $t \to \infty$. It is clear from inspection of (3.6) and (3.7) that as $t \to \infty$, $d \sim Vt/\lambda$ and $a_n \sim 1 - A_n\,e^{-\delta t}$, where A_n are positive constants depending on the λ_n and $a_n(0)$, and where $\delta = 1/2\lambda(1-\lambda)$. The base of the finger near $y = \pi$ is the image under (3.5) of the segment $-\pi \leqslant \operatorname{Arg}\zeta \leqslant -\pi + O(e^{-\delta t})$ of $|\zeta| = 1$, and setting $\zeta = -1 - i\tau\,e^{-\delta t} + o(e^{-\delta t})$ where τ is $O(1)$, we obtain parametric equations for the limiting form of the moving boundary as

$$\left.\begin{aligned} x &= \text{constant} + \frac{1}{N}\sum_{n=1}^{N}(1-\lambda_n)\ln(A_n^2+\tau^2), \\ y &= \pi - \frac{1}{N}\sum_{n=1}^{N} 2(1-\lambda_n)\arctan\left(\frac{\tau}{A_n}\right) \end{aligned}\right\} \quad \text{for } 0 \leqslant \tau < \infty, \tag{3.8}$$

with $\tau \to \infty$ being the flat side of the finger $x \to \infty$, $y \to \lambda\pi$ (which in coordinates moving with the finger tip is the downstream asymptote). The shape of the curve (3.8) can be varied by choosing different A_n and λ_n, and we note in particular that we must have at least one of the $\lambda_n > 1$ if we seek a curve which is not monotone in y (see the examples below).

The figures used below were produced by rewriting (3.6) and (3.7) in terms of the A_n rather than the a_n, since the latter become extremely close to 1. The only real computation involved was the solution of equations (3.6), (3.7) at each time at which a figure was required, which was performed using a library routine. In all the figures, $V = 1$ and the moving boundary is shown for $t = 0, 1, 2, \ldots, 9$; the x- and y-axes have been omitted for clarity.

Example 3.1 (figure 1*a*) $N = 1$, $a_1(0) = 0.05$, $\lambda_1 = 0.5$. This is Saffman's solution (1959) with $\lambda = \frac{1}{2}$; the initial interface is $x = 0.05\cos y + O(10^{-3})$.

Example 3.2 (figure 1*b*) $N = 2$, $a_1(0) = 0.01$, $a_2(0) = 0.11$, $\lambda_1 = 0.4$, $\lambda_2 = 0.6$, $\lambda = 0.5$. Again the initial interface is $x = 0.05\cos y + O(10^{-3})$, but now the base of the finger develops a 'hump' as $t \to \infty$, although the behaviour near the tip for large times is as in example 3.1.

Example 3.3 (figure 1*c*) $N = 2$, $a_1(0) = 0.2$, $a_2(0) = 0.6$, $\lambda_1 = 1.8$, $\lambda_2 = -0.2$, $\lambda = 0.8$. The noteworthy feature of this example is that it forms a 'neck' at its base (note that one of the λ_n is greater than 1 here, giving a negative coefficient in (3.5). This phenomenon is seen in practice (Taylor & Saffman 1958) and in numerical simulations of Hele-Shaw flow (Tryggvason & Aref 1985) although in reality surface tension and other effects which we have ignored will predominate near the base where the velocities are small, so that we do not expect our solutions to give accurate representations of the behaviour there. It is nevertheless interesting that we can obtain this behaviour without recourse to these extra hypotheses, a point which also applies to our next example.

Example 3.4 (figure 1*d*) $N = 3$, $a_1(0) = 0.2$, $a_2(0) = 0.04$, $a_3(0) = 0.4$, $\lambda_1 = 2.65$, $\lambda_2 = -0.05$, $\lambda_3 = -0.8$; $\lambda = 0.6$. In this example, the extra term (from $N = 3$ instead of 2), allows a 'secondary finger' to grow at the base. This behaviour is reminiscent of secondary dendritic growth in the solidification of a binary alloy or a supercooled liquid (Langer 1980), and although again we do not claim that our simple model will accurately represent real secondary dendritic growth, because it ignores diffusion and surface energy effects, it is again noteworthy that this kind of behaviour *can* occur even in the simple model. Similar secondary growths have been observed in the numerical simulations of Kessler, Koplik & Levine (1985), albeit in a radial geometry (but we can produce these secondary growths there too, see §4). Our example shows

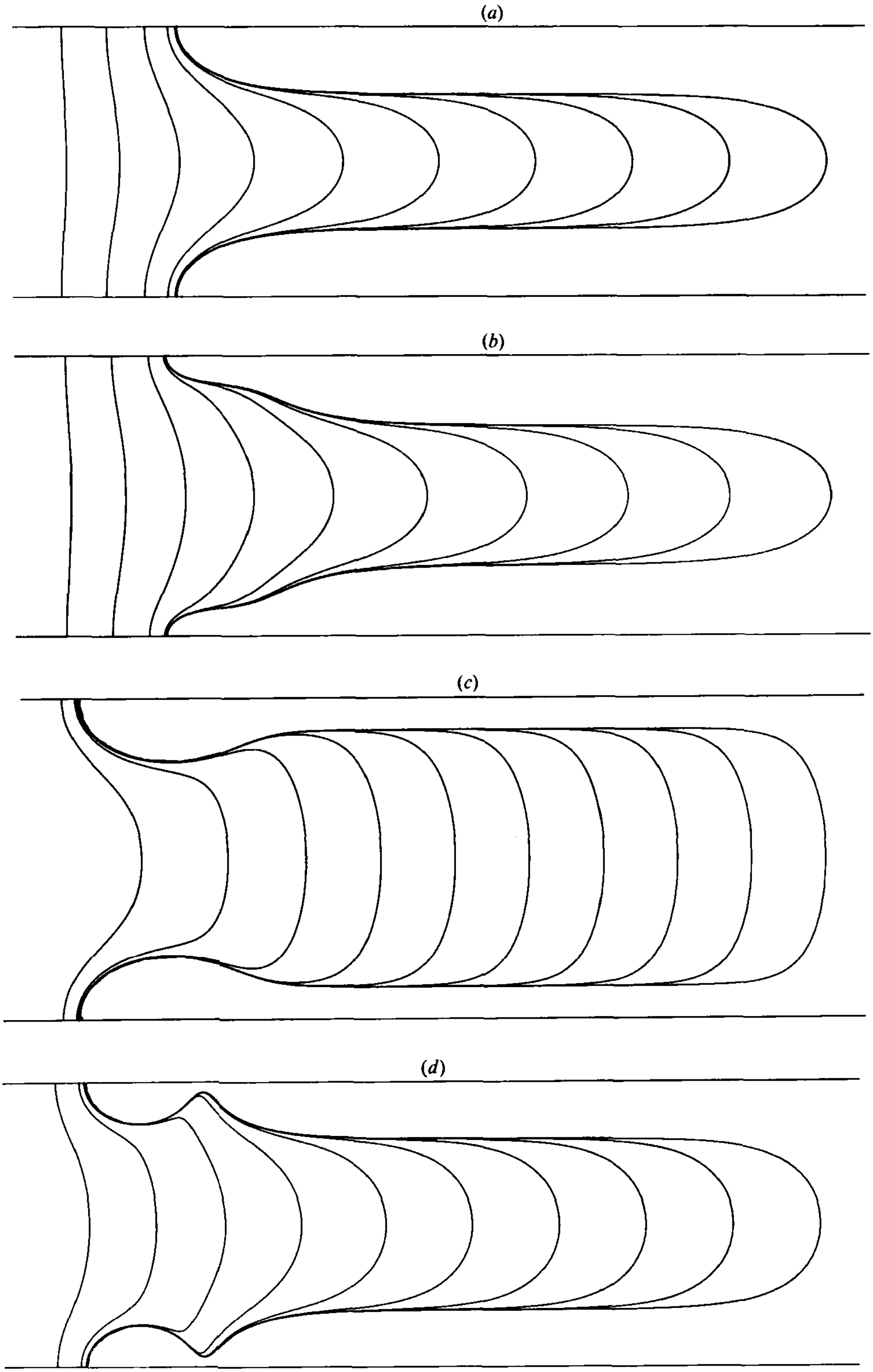

FIGURE 1. Fingering in a channel: (a) Saffman's solution; (b) with a 'hump'; (c) with a 'neck'; (d) with a secondary finger.

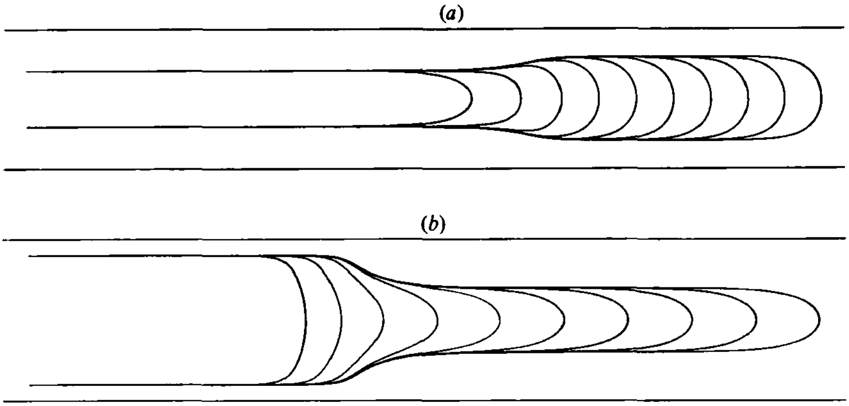

FIGURE 2. Fingers changing width in a channel.

that anisotropic surface tension is not – as they conjecture – necessary for secondary growth.

More complicated base shapes, including any finite number of secondary fingers, can be created by taking more terms in (3.5) and choosing the λ_n and $a_n(0)$ appropriately. In particular, we may obtain one extra secondary finger every time we increase N by 2, although whether we do or not depends on the precise values of $\lambda_n, a_n(0)$. (If we desire a base shape with a given number of turning points it is easiest to choose N, A_n and λ_n from (3.8) and to run the problem backwards in time).

We can obtain more solutions by relaxing the restriction $0 \leqslant a_n < 1$ in (3.5). A particularly interesting example is to take $N = 2, a_1 \equiv 1, 0 < a_2(0) \ll 1$. If $a_2(0)$ had been zero, this solution would represent a travelling-wave finger (3.3) with initial width $\lambda_i = \frac{1}{2}(1+\lambda_1)$, and so the small term $(1-\lambda_2)\ln(1+a_2(0)\zeta)$ represents a small initial perturbation on a travelling-wave finger. At $t \to \infty$, however, $a_2 \to 1$, and the solution tends to another travelling-wave finger with final width $\lambda_f = \frac{1}{2}(\lambda_1+\lambda_2)$. With suitable λ_1 and λ_2 the finger can be made either wider or narrower by this perturbation, as in the next two examples.

Example 3.5 (figure 2*a*) $N = 2, a_1 \equiv 1, a_2(0) = 0.05, \lambda_1 = -0.2, \lambda_2 = 1.8\,(\lambda_i = 0.4, \lambda_f = 0.8)$: this finger becomes broader.

Example 3.6 (figure 2*b*) $N = 2, a_1 \equiv 1, a_2(0) = 0.05, \lambda_1 = 0.6, \lambda_2 = 0.2\,(\lambda_i = 0.8, \lambda_f = 0.4)$: this finger becomes narrower.

For our final example of developing fingers, we realise that it is not in fact necessary to assume that all the singular points of $f(\zeta;t)$ lie on the negative real axis. Indeed, we can consider a generalization of (3.5) of the form

$$z = f(\zeta;t) = d(t) - \ln\zeta + \sum_{n=1}^{N} [\beta_n \log(1+\alpha_n(t)\zeta) + \bar{\beta}_n \log(1+\overline{\alpha_n(t)}\zeta)], \tag{3.9}$$

in which the β_n are complex constants and the α_n are complex functions of t with $|\alpha_n| < 1$ (the complex-conjugate terms ensure that the finger is symmetrical); we take the branch cuts to lie along rays from $\zeta = -1/\alpha_n, -1/\overline{\alpha_n}$ to infinity. Differentiating and clearing fractions we see that $\zeta\partial f/\partial\zeta\,\overline{\partial f/\partial t}$ is of the form

$$P_{2N}(\zeta)\,Q_{2N}(\bar{\zeta})/\prod_{n=1}^{N}|1+\alpha_n\zeta|^2|1+\overline{\alpha_n}\zeta|^2,$$

where P_{2N} and Q_{2N} are polynomials of degree $2N$ with real leading coefficients. When

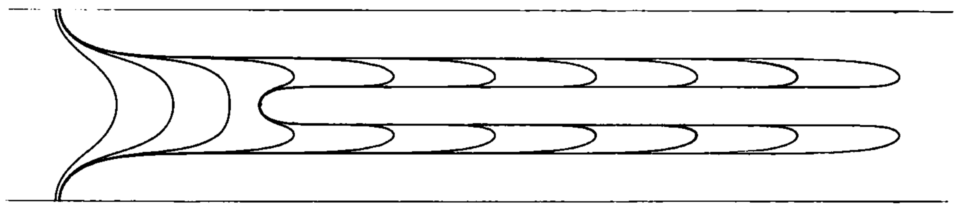

FIGURE 3. Asymmetrical fingering in a channel.

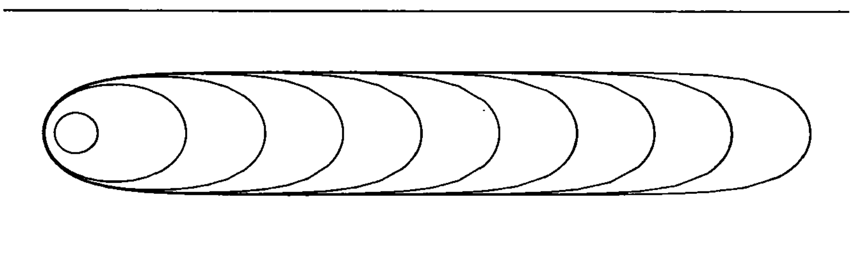

FIGURE 4. Growth of a bubble in a channel.

we substitute into (2.4), therefore, and equate coefficients of powers of ζ and $\bar{\zeta}$, we will obtain N complex and one real ordinary differential equations for the α_n and d: the real equation will describe rate of change of area, while we can integrate the complex ones by the procedure of §5. Equation (3.9) thus gives a very general class of solutions to the Hele-Shaw problem in a channel. The details are complicated in that we must choose starting values to ensure that $f' \neq 0$ and that f is $1-1$ for all t, and it is in fact possible to have cusps for $N > 1$ with an injudicious choice of starting values, but we will not consider such solutions here (see Howison 1986*a* and Bensimon *et al.* 1985). Instead we give the example

$$z = d(t) - \ln \zeta + \beta_1 \log(1+\alpha_1 \zeta) + \beta_2 \log(1-\alpha_2 \zeta), \tag{3.10}$$

in which $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real and positive. Starting with $0 < \alpha_2 \ll \alpha_1 \ll 1$, we see an initially almost-harmonic perturbation of a planar interface which initially grows like a single finger (3.4), but at later times splits into two fingers symmetrically disposed above and below the centreline of the channel. These two fingers are themselves asymmetrical, and provided that $0 < \beta_1 + \beta_2 < 2, \beta_1 \neq \beta_2$, we can ensure that their limiting form near the tip is also asymmetrical (see figure 3, in which $\beta_2 = 1$, $\beta_2 = 0.4, \alpha_1(0) = 0.4, \alpha_2(0) = 0.001$). These asymmetrical travelling-wave limits were noted by Taylor & Saffman (1959).

In our final example of channel flow we return to (3.5) and take $N = 1$, but we now allow a_1 to be greater than 1. In this case $f(\zeta; t)$ has another singularity at $\zeta = -1/a_1$ inside $|\zeta| < 1$, and we must choose $\lambda = \frac{1}{2}$ so that we can take the branch cut for the logarithm terms between $\zeta = 0$ and $\zeta = -1/a_1$. The top and bottom of this cut are mapped onto the channel walls, and the unit circle is mapped onto the boundary of a bubble which grows as fluid is extracted from $x = +\infty$: see figure 4, in which $a_1(0) = 2$. This solution is similar to that of Jacquard & Séguier (1962), except that in their solution fluid is extracted from both $x = +\infty$ and $x = -\infty$, and their bubble is symmetrical. Again, near the upstream tip of the bubble, we retrieve the travelling wave (3.3) as $t \to \infty$.

In the next section we turn from bubbles in channels to bubbles in infinite expanses of fluid without walls.

4. Radial fingering

The second configuration that we discuss, and the only other for which any experimental evidence is, as far as we know, available, is that of a bubble growing by injection of air in an infinite expanse of fluid without walls. Using the same complex-variable technique as above we can construct classes of solutions analogous to those given in §3 for channel flow, and we can make a qualitative comparison with the published experimental results of Paterson (1981).

The appropriate form for the conformal map from the unit disk onto $\Omega(t)$ is now

$$z = f(\zeta; t) = \frac{a(t)}{\zeta} + k(\zeta; t), \tag{4.1}$$

where again f is conformal in $0 < |\zeta| < 1$, and where $a(t)$ is real and positive so that there is no rotation at infinity; again f must satisfy (2.12).

Some results are known for this version of the problem. These we summarize; our interest is chiefly in new examples of 'fingering' solutions which exist for all time. Suppose that we consider an initial-value problem: we are given $\Omega(0)$, the shape of the bubble at $t = 0$. We distinguish between three possible types of behaviour:

(*a*) The solution may blow-up, typically via a cusp, after finite time. This can be shown to be the case whenever $k(\zeta; 0)$ is a polynomial of finite degree (Howison 1986*a*; see also the brief discussion by Shraiman & Bensimon 1984), and thus our remark that there is a large number of 'bad' solutions close to any 'good' one holds in this configuration too. (On the other hand, it is possible for a cusp of a different kind (a $\frac{5}{2}$-power rather than a $\frac{3}{2}$-power) to form in $\partial\Omega$ *without* blow-up; in view of the impossibility of realizing these cusps in practice we shall not discuss this point here, but refer to Howison 1986*a*.)

(*b*) The solution may exist for all time and, as the area of the bubble tends to infinity, it may exhaust the whole space, in the sense that $\partial\Omega(t)$ eventually crosses every point in $\Omega(0)$. It can be shown that this exhaustion can only be achieved if $\partial\Omega(0)$ is an ellipse, and in this case $\partial\Omega(t)$ remains an ellipse of the same eccentricity for all t (Howison 1986*b*, DiBenedetto & Friedman 1985).

(*c*) The third possible outcome, and the one which is seen in practice unless the bubble grows extremely slowly (when case (*b*) may apply because surface tension will stabilize small perturbations in the boundary) is that the bubble may evolve a shape which, as $t \to \infty$, leaves some 'tongues', or even isolated blobs of fluid behind.

The main result of this section is the presentation of a large class of solutions with this behaviour.

Guided by the solutions of §3, we try a combination of logarithms for $k(\zeta; t)$. We find in fact that a solution exists for any $k(\zeta; t)$ of the form

$$k(\zeta; t) = \sum_{n=1}^{N} \beta_n \log\left(\zeta - c_n(t)\right), \tag{4.2}$$

where $c_n(t), \beta_n$ are complex (the argument is the same as that which leads to (3.9)), but again the details are too complicated to make a full study of this equation worthwhile. The most general form which we shall consider is

$$z = \frac{a(t)}{\zeta} + \beta_1 \sum_{k=1}^{N} \omega^{-k} \log\left(c_1(t)\,\omega^k - \zeta\right) + \beta_2 \sum_{k=1}^{N} \omega^{-k-\frac{1}{2}} \log\left(c_2(t)\,\omega^{k+\frac{1}{2}} - \zeta\right), \tag{4.3}$$

where a, c_1 and c_2 are real functions of time with $c_1, c_2 > 1$, β_1 and β_2 are positive constants at our disposal, $N \geqslant 1, \omega^N = 1$, and the branch cuts for the logarithms are taken radially outwards from their singular points to infinity.

Substitution into (2.12) verifies that (4.3) is in fact an explicit solution of the Hele-Shaw problem provided that $f(\zeta;t)$ is conformal, and we obtain the equations

$$a^2 - Na(\beta_1 c_1 + \beta_2 c_2) = \frac{Qt}{\pi} + K_0, \tag{4.4}$$

$$ac_1 + \beta_i \sum_{k=1}^{N} \omega^{-k} \log(c_i^2 \omega^k - 1) + \beta_j \sum_{k=1}^{N} \omega^{-k-\frac{1}{2}} \log(c_i c_j \omega^{k+\frac{1}{2}} - 1) = K_i, \tag{4.5}$$

where (4.5) holds for $i = 1, j = 2$ and for $j = 1, i = 2$; K_0, K_1 and K_2 are real constants of integration determined by $a(0)$, $c_1(0)$ and $c_2(0)$. The first of these equations is the rate of change of area for the bubble, while (4.5) is obtained by the procedure of §5.

Suppose that we start with $c_1(0)$ and $c_2(0)$ large; $\partial\Omega(0)$ is nearly circular, since then $f(\zeta;0) \sim a(0)/\zeta + O(\zeta/c_1, \zeta/c_2)$. As t decreases, c_1 and c_2 decrease, tending to 1 as $t \to \infty$, and as singularities of the logarithmic terms in (4.2) approach $|\zeta| = 1$, so the bubble develops $2N$ fingers of air leaving $2N$ 'tongues' of fluid behind (see figure 5). These tongues are arranged so that the N corresponding to the singularities at $\zeta = c_1 \omega^k$ occur alternately with those corresponding to $\zeta = c_2 \omega^{k+\frac{1}{2}}$. The parameters β_1 and β_2 are equivalent to $1-\lambda$ in a channel flow, in that the width of the tongues is $\pi\beta_1$ or $\pi\beta_2$ corresponding to $\zeta \sim c_1 \omega^k$, $c_2 \omega^{k+\frac{1}{2}}$ respectively.

It is also worth noting that N is at our disposal. The number of fingers that will be seen in practice to emerge from a growing bubble (as, indeed, the number of fingers growing out of a nearly planar interface in a very broad channel) is determined by stability criteria involving *inter alia* surface-tension effects, which fall outside the scope of our simple model (see Paterson 1981). Nevertheless once N is chosen for us in this way, we expect that for small surface tension our solutions should be in some sense 'near' to those seen in practice.

The moving boundary has an asymptotic shape as $t \to \infty$ which consists of a set of $2N$ tongues with parallel sides as $r \to \infty$, and with tips at $z = K_1 \omega^k$, $K_2 \omega^{k+\frac{1}{2}}$ $(k = 0, 1, \ldots, N-1)$. The shape of these tongues is the same as the asymptotic shapes of the channel solutions of §3, which follows from the similarity between the mapping functions. The tips of the tongues are at our disposal through $a(0), c_1(0), c_2(0)$, and we can, for example, choose $K_2 \gg K_1$ $(c_2(0) \gg c_1(0))$ so that the influence of the singularities at $\zeta = c_2(t)\omega^{k+\frac{1}{2}}$ is not felt until the tongue corresponding to the singularities at $\zeta = c_1(t)\omega^k$ is well developed. In this case the solution mimics the 'finger-splitting' behaviour observed by Paterson (1981): see, for example, the finger in the fourth quadrant of figure 6 (although this is more asymmetrical than any finger we can produce with the two sums in (4.2)), figure 9 of Paterson's paper, and example 4.2 below.

We give two examples; $Q = 2\pi$ in each case.

Example 4.1 (figure 5*a*) $N = 3$, $\beta_1 = \beta_2 = 2$, $a_1(0) = 10$, $c_1(0) = c_2(0) = 1.2$. The initial interface is nearly circular, and six identical fingers develop. The time interval Δt between successive pictures of the interface is 100.

Example 4.2 (figure 5*b*) $N = 6$, $\beta_1 = 3$, $\beta_2 = 2$, $a(0) = 10$, $c_1(0) = 2$, $c_2(0) = 1.1$, $\Delta t = 60$. Here the interface starts with six already pronounced fingers which soon split into twelve asymmetrical fingers.

A detailed comparison between these solutions and the experimental results of Paterson (1981) would require a more general form for $k(\zeta;t)$ than (4.2) in order to reproduce the asymmetries seen in his experiments. Nevertheless, the similarity

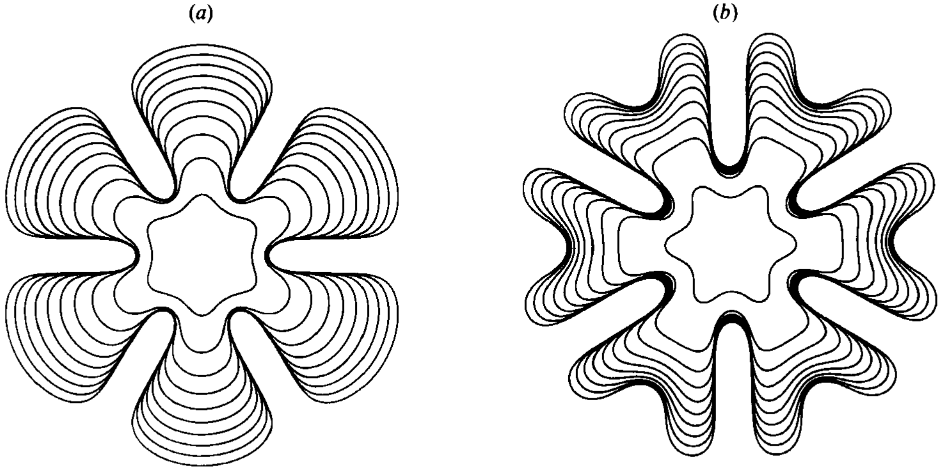

FIGURE 5. Radial fingering.

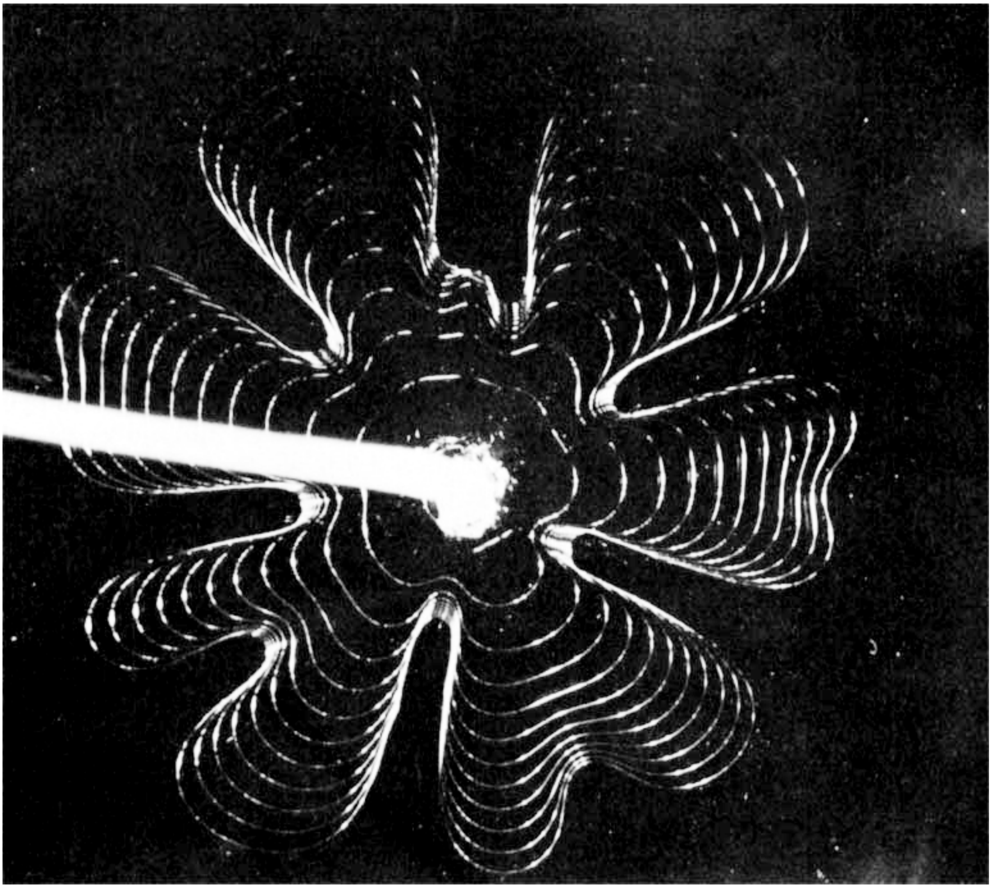

FIGURE 6. A multiple-exposure photograph of an air bubble growing in a Hele-Shaw cell filled with glycerine (kindly supplied by Dr L. Paterson).

between one of the fingers in example 4.1 and the finger in the first quadrant of figure 6 (which is a photograph of an expanding bubble kindly supplied by Dr L. Paterson; details of the experiment are given in Paterson 1981) are striking. A general observation is that the tips of the experimental fingers tend to be more rounded than the tips in these particular exact solutions, and that their bases do move outwards, albeit very slowly: both of these effects may most likely be attributable to factors such as surface tension at the moving boundary, which we have ignored.

Lastly we point out again that many more complex fingering patterns are attainable by taking a more complicated form for $k(\zeta;t)$. For example, by adding sums of logarithms with branch points at $\zeta = c_3(t)\,\omega^{k+\frac{1}{4}}$, $\zeta = c_4(t)\,\omega^{k+\frac{3}{4}}$, we can make each finger split twice; by adding more sums of logarithms with branch points on the lines $\arg\zeta = 2k\pi/N$, $\arg\zeta = (2k+1)\pi/2N$, we can achieve the same behaviour at the bases of the fingers as we did in §3 in channels: clearly many other combinations could be made. The complete classification of solutions which do not blow up in finite time remains an open question, although DiBenedetto & Friedman (1985) prove a result that, given any asymptotic set of tongues of fluid (that is, a set of curves representing the boundary of the fluid after an infinite time has elapsed), and subject to certain symmetry and growth conditions on these tongues, one can prove the existence and uniqueness of a Hele-Shaw bubble solution (in the weak sense) tending to this set as $t \to \infty$. Unfortunately the conditions of the theorem are too restrictive to apply to our solutions, since one of the growth requirements is that the tongue thickness should tend to infinity as $r \to \infty$.

5. The Schwarz Function for $\partial\Omega$

The differential equations that we obtain for the functions of time in our solutions are complicated (except for the one which expresses conservation of mass), and it is fortunate that we are able to write their solution without having to solve them directly. The procedure relies on a transformation of the dependent variable introduced by Lacey (1982). We first suppose that we can solve for the moving boundary in the form $t = \omega(x,y)$ for each t: successive moving boundaries are thus level curves of the function $\omega(x,y)$, which is defined at all points crossed by the moving boundary in $0 \leqslant t < \infty$ (or while the solution exists). We shall call this region R. We now define

$$u(x,y,t) = -\int_t^{\omega(x,y)} p(x,y,\tau)\,\mathrm{d}\tau \tag{5.1}$$

$$= u(x,y,0) + \int_0^t p(x,y,\tau)\,\mathrm{d}\tau \tag{5.2}$$

for $(x,y) \in R$. Direct differentiation shows that

$$\nabla^2 u = 1 \quad \text{in } \Omega(t) \cap R, \tag{5.3}$$

with

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega(t); \tag{5.4}$$

note also that

$$\frac{\partial u}{\partial t} = p \quad \text{in } \Omega(t) \cap R. \tag{5.5}$$

Equation (5.3) allows analytic continuation of u into $\Omega(t)\backslash R$ for each t, although such a continuation will in general have singularities. However, since p is without singularities in $\Omega(t)\backslash\{\infty\}$, (5.5) shows that the singularities of u in $\Omega(t)\backslash\{\infty\}$ must be those of $u(x, y, 0)$ only, and that they are constant in time. (It is possible for the analytic continuation of u *outside* $\Omega(t)$, i.e. into the air, to have moving singularities, and indeed any finite-time blow-up is usually associated with one of these moving singularities reaching $\partial\Omega(t)$ and causing a cusp. If the moving boundary were to cross one of the *internal* singularities of u there would also be a loss of analyticity in $\partial\Omega(t)$; however, no examples of this behaviour are known, nor even whether it is possible.)

The observation which permits us to integrate our differential equations is that there is a connection between u and the mapping function $f(\zeta; t)$ which generates $\partial\Omega(t)$. First we note that $\partial\Omega(t)$, being an analytic curve, can be written in the form

$$\bar{z} = g(z; t), \tag{5.6}$$

where g is analytic in a neighbourhood of $\partial\Omega(t)$ for each t: g is called the Schwarz function of $\partial\Omega$ (Davis 1974).

Now consider $u - \frac{1}{4}z\bar{z}$, which is harmonic: $u_x - \mathrm{i}u_y - \frac{1}{2}\bar{z}$ is thus an analytic function which, using (5.5) and (5.6), is equal to $-\frac{1}{2}g(z; t)$ on $\partial\Omega(t)$. Hence, using the identity theorem for analytic functions,

$$u_x - \mathrm{i}u_y = \tfrac{1}{2}(\bar{z} - g(z; t)) \tag{5.7}$$

wherever either exists. This result is due to Lacey (1982). Equation (5.7) shows that the singular points of u and those of g are one and the same, and in particular that the singular points of g inside $\Omega(t)$ must be constant in time (and that if the solution exists for all time without developing cusps, the internal singularities of g must remain internal, since they cannot be crossed by $\partial\Omega(t)$).

The procedure is thus to look at the internal singularities of g, via the equation

$$g(z(\zeta; t)) = \hat{g}(\zeta; t) = \overline{f(1/\bar{\zeta}; t)} = \bar{f}(\zeta; t) \tag{5.8}$$

($\hat{g}$ is the function which is equal to $\bar{f}(\zeta; t) = \bar{z}$ on $|\zeta| = 1$), and to use the fact that they are constant in time. In all our examples we have just a sufficient number of singularities of g inside $|\zeta| = 1$ that we can integrate all the equations we need to. For example, in (3.4), $\bar{f}(\zeta; t)$ has a branch point at $\zeta = -a(t)$, corresponding to

$$z = d(t) - \ln a(t) + 2(1-\lambda) \ln \tfrac{1}{2}(1 - a^2(t)) \pm \mathrm{i}\pi,$$

and this gives (3.7) ($N = 1$) immediately. We also observe that whenever a 'tongue' of fluid is formed there is a singularity of g at its tip, so that if we wished to construct a solution ending with a given number of tongues in specified positions we could attempt to do so by constructing a g with the appropriate singularities. This is, however, unsatisfactory in that the construction can only be carried out *post facto* and not in a predictive way. The question of exactly how surface tension 'selects' a particular solution from the many candidates (both blowing-up and existing for all time) remains a challenging problem.

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